Lecture 3

Interpolation

Suppose we are given the following values of $y = f(x)$ for a set of values of x:

x:	x_0	x_1	$x_2 \cdots x_n$
y:	Y_0	y_1	$y_2 \cdots y_n$.

Then the process of finding the value of y corresponding to any value of $x = x_i$ between x_0 and x_n is called *interpolation*. Thus *interpolation is the technique of estimating the value of a function for any intermediate value of the independent variable* while the process of computing the value of the function outside the given range is called *extrapolation*. The term interpolation however, is taken to include extrapolation.

If the function $f(x)$ is known explicitly, then the value of y corresponding to any value of x can easily be found. Conversely, if the form of $f(x)$ is not known (as is the case in most of the applications), it is very difficult to determine the exact form of $f(x)$ with the help of tabulated set of values (x_i, y_i). In such cases, $f(x)$ is replaced by a simpler function $\phi(x)$ which assumes the same values as those of $f(x)$ at the tabulated set of points. Any other value may be calculated from $\phi(x)$ which is known as the *interpolating function* or *smoothing function*. If $\phi(x)$ is a polynomial, then it called the *interpolating polynomial* and the process is called the *polynomial interpolation*. Similarly when $\phi(x)$ is a finite trigonometric series, we have trigonometric interpolation. But we shall confine ourselves to polynomial interpolation only.

In the following we will learn how to get this polynomial representation for these descrete points.

The Lagrange interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where n *in* $f_n(x)$ *stands for the* n^{th} *order polynomial that approximates the function* $y = f(x)$ *given at* $n+1$ *data points as* $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$*, and*

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$ *is a weighting function that includes a product of* $n-1$ *terms with terms of* $j = i$ *omitted.*
The application of Lagrange interpolation will be clarified using an example.

Example 1

Given the values

x	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

evaluate f(9), using Lagrange's formula
Solution:
Here given 5 points so the largest polynomial that we can get is from degree 4

$$P_4(x) = \sum_{i=0}^{4} L_i(x)\, y_i \ \ \& \ L_i(x) = \prod_{\substack{k=0 \\ k \neq i}}^{n} \frac{(x - x_k)}{(x_i - x_k)}, n \text{ is no. of used points}$$

$$P_4(x) = L_0(x)\, y_0 + L_1(x)\, y_1 + L_2(x)\, y_2 + L_3(x)\, y_3 + L_4(x)\, y_4$$

$$\begin{aligned} P_4(x) = {} & \frac{(x - x_1)(x - x_2)(x - x_3)(x - x_4)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)(x_0 - x_4)} \times y_0 \\ & + \frac{(x - x_0)(x - x_2)(x - x_3)(x - x_4)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} \times y_1 \\ & + \frac{(x - x_0)(x - x_1)(x - x_3)(x - x_4)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} \times y_2 \\ & + \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_4)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} \times y_3 \\ & + \frac{(x - x_0)(x - x_1)(x - x_2)(x - x_3)}{(x_4 - x_0)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} \times y_4 \end{aligned}$$

Putting $x = 9$ and substituting the above values in Lagrange's formula, we get

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$
$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452$$
$$+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$
$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$
$$= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} + \frac{2366}{3} + \frac{578}{5} = 810$$

Example 2

Find the polynomial $f(x)$ by using Lagrange's formula and hence find $f(3)$ for

x:	0	1	2	5
$f(x)$:	2	3	12	147

Solution:

Here $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 5$

and $y_0 = 2, y_1 = 3, y_2 = 12, y_3 = 147.$

Lagrange's formula is

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$
$$= \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$
$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

Hence $f(x) = x^3 + x^2 - x + 2$

$\therefore \quad f(3) = 27 + 9 - 3 + 2 = 35$

Comment: also we can get the value of $f'(x) = 3x^2 + 2x - 1$ *and* $f'(3) = 3(3^2) + 2(3) - 1 = 32.$

Example: 3

A curve passes through the points (0, 18), (1, 10), (3, –18) and (6, 90). Find the slope of the curve at $x = 2$.

Solution:

Here $x_0 = 0, x_1 = 1, x_2 = 3, x_3 = 6$ and $y_0 = 18, y_1 = 10, y_2 = -18, y_3 = 90$.

Since the values of x are unequally spaced, we use the Lagrange's formula:

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$
$$= \frac{(x-1)(x-3)(x-6)}{(0-1)(0-3)(0-6)}(18) + \frac{(x-0)(x-3)(x-6)}{(1-0)(1-3)(1-6)}(10)$$
$$+\frac{(x-0)(x-1)(x-6)}{(3-0)(3-1)(3-6)}(-18) + \frac{(x-0)(x-1)(x-3)}{(6-0)(6-1)(6-3)}(90)$$
$$= (-x^3 + 10x^2 - 27x + 18) + (x^3 - 9x^2 + 18x)$$
$$+(x^3 - 7x^2 + 6x) + (x^3 - 4x^2 + 3x)$$

i.e., $y = 2x^3 - 10x^2 + 18$

Thus the slope of the curve at $x = 2 = \left(\frac{dy}{dx}\right)_{x-2}$

$$= (6x^2 - 20x)_{x=2} = -16$$

Example 4

Find the missing term in the following table using interpolation:

x:	0	1	2	3	4
y:	1	3	9	...	81

Solution:

Since the given data is unevenly spaced, therefore we use Lagrange's interpolation formula:

$$y = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Here we have $x_0 = 0 \quad x_1 = 1 \quad x_2 = 2 \quad x_3 = 4$

$y_0 = 1 \quad y_1 = 3 \quad y_2 = 9 \quad y_3 = 81$

$$\therefore \quad y = \frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)}(1) + \frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}(3)$$
$$+ \frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)}(9) + \frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)}(81)$$

When $x = 3$, then

$$\therefore \quad y = \frac{(3-1)(3-2)(3-4)}{-8} + 3(3-2)(3-4) + \frac{3(3-1)(3-4)(9)}{-4} +$$
$$+ \frac{3(3-1)(3-2)}{24}(81) = \frac{1}{4} - 3 + \frac{27}{2} + \frac{81}{24} = 31$$

Hence the missing term for $x = 3$ is $y = 31$.

Example 5

Find the distance moved by a particle and its acceleration at the end of 4 seconds, if the time verses velocity data is as follows:

t:	0	1	3	4
v:	21	15	12	10

Solution:

Since the values of t are not equispaced, we use Lagrange's formula:

$$v = \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_0-t_1)(t_0-t_2)(t_0-t_3)}v_0 + \frac{(t-t_0)(t-t_2)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}v_1$$
$$+ \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}v_2 + \frac{(t-t_0)(t-t_1)(t-t_3)}{(t_1-t_0)(t_1-t_2)(t_1-t_3)}v_3$$

$$.e. \ , v = \frac{(t-1)(t-3)(t-4)}{(-1)(-2)(-4)}(21) + \frac{t(t-3)(t-4)}{(1)(-2)(-3)}(15)$$
$$+ \frac{t(t-1)(t-4)}{(3)(2)(-1)}(12) + \frac{t(t-1)(t-3)}{(4)(3)(1)}(10)$$

$$.e., \ v = \frac{1}{12}(-5t^3 + 38t^2 - 105^t + 252)$$

$$\therefore \text{ Distance moved } s = \int_0^4 v\,dt = \int_0^4 (-5t^3 + 38t^2 - 105^t + 252) \qquad \left[\because v = \frac{ds}{dt}\right]$$

$$= \frac{1}{12}\left(-\frac{5t^4}{4} + \frac{38t^3}{3} - \frac{105t^2}{2} + 252t\right)_0^4$$

$$= \frac{1}{12}\left(-320 + \frac{2432}{3} - 840 + 1008\right) = 54.9$$

$$\text{Also acceleration} \qquad = \frac{dv}{dt} = \frac{1}{2}(-15t2 + 76t - 105 + 0)$$

$$\text{Hence acceleration at } (t = 4) = \frac{1}{2}(-15 \pm +76(4) - 105) = -3.4$$

Newton's Divided Difference Polynomial Method

To illustrate this method, linear and quadratic interpolation is presented first. Then, the general form of Newton's divided difference polynomial method is presented. To illustrate the general form, cubic interpolation is shown in Figure 1.

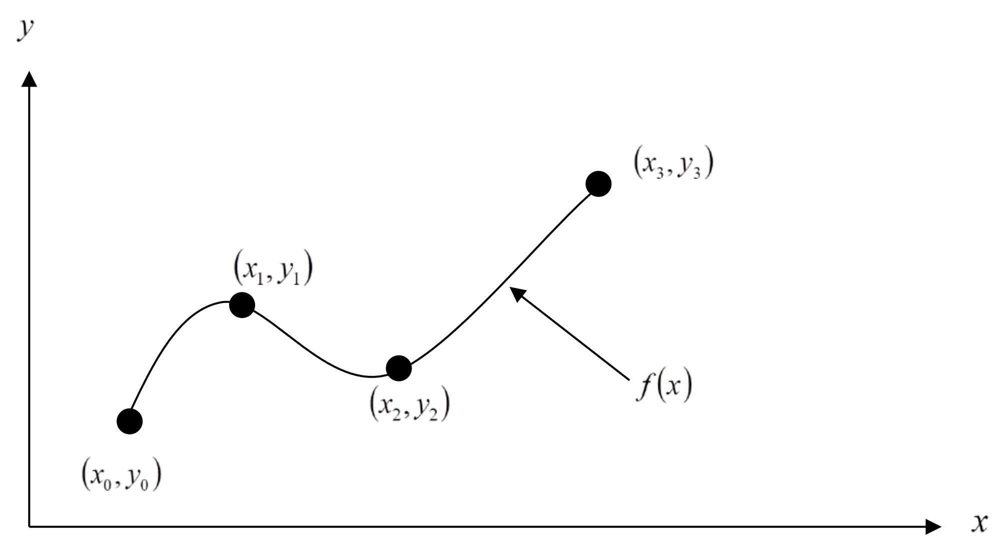

Figure 1 *Interpolation of discrete data.*

Linear Interpolation

Given (x_0, y_0) *and* (x_1, y_1), *fit a linear interpolant through the data. Noting* $y = f(x)$ *and* $y_1 = f(x_1)$, *assume the linear interpolant* $f_1(x)$ *is given by (Figure 2)*

$$f_1(x) = b_0 + b_1(x - x_0)$$

Since at $x = x_0$,

$$f_1(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) = b_0$$

and at $x = x_1$,

$$f_1(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0)$$
$$= f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

So

$$b_0 = f(x_0)$$
$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

giving the linear interpolant as

$$f_1(x) = b_0 + b_1(x - x_0)$$
$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

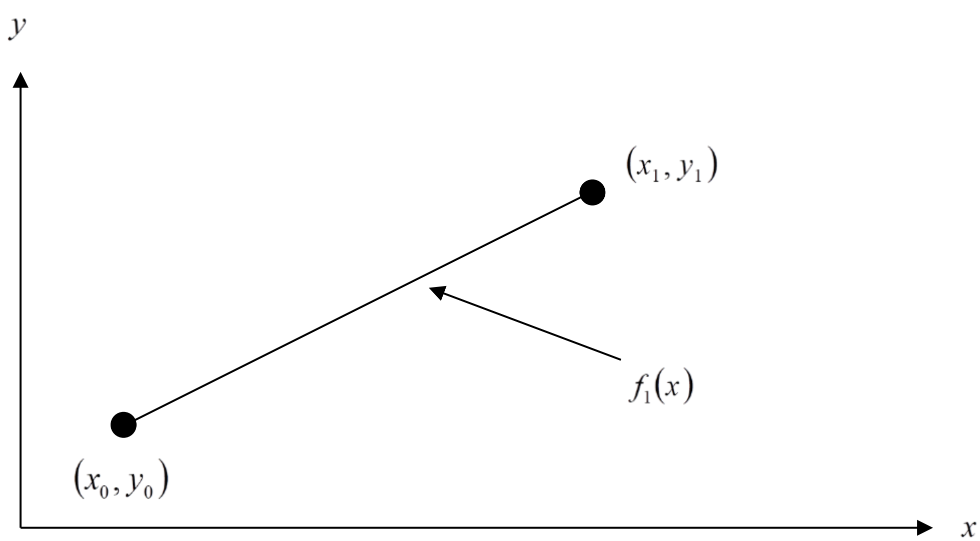

Figure 2 *Linear interpolation.*

Quadratic Interpolation

Given (x_0, y_0), (x_1, y_1), *and* (x_2, y_2), *fit a quadratic interpolant through the data. Noting* $y = f(x)$, $y_0 = f(x_0)$, $y_1 = f(x_1)$, *and* $y_2 = f(x_2)$, *assume the quadratic interpolant* $f_2(x)$ *is given by*

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

At $x = x_0$,

$$f_2(x_0) = f(x_0) = b_0 + b_1(x_0 - x_0) + b_2(x_0 - x_0)(x_0 - x_1)$$
$$= b_0$$
$$b_0 = f(x_0)$$

At $x = x_1$

$$f_2(x_1) = f(x_1) = b_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1)$$

$$f(x_1) = f(x_0) + b_1(x_1 - x_0)$$

giving

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

At $x = x_2$

$$f_2(x_2) = f(x_2) = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

Giving

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Hence the quadratic interpolant is given by

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) + \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}(x - x_0)(x - x_1)$$

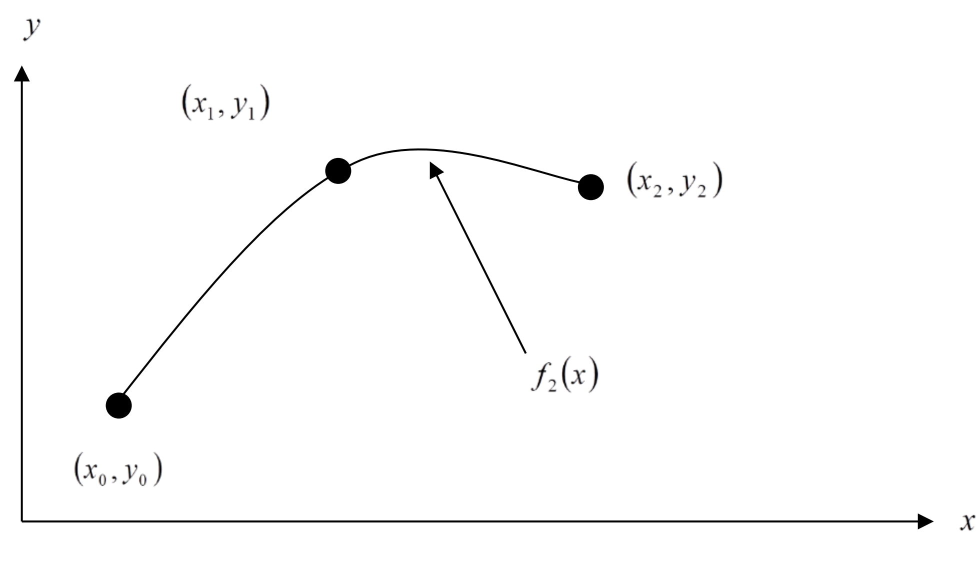

Figure 4 *Quadratic interpolation.*

General Form of Newton's Divided Difference Polynomial

In the two previous cases, we found linear and quadratic interpolants for Newton's divided difference method. Let us revisit the quadratic polynomial interpolant formula

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Note that b_0, b_1, *and* b_2 *are finite divided differences.* b_0, b_1, *and* b_2 *are the first, second, and third finite divided differences, respectively. We denote the first divided difference by*

$$f[x_0] = f(x_0)$$

the second divided difference by

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and the third divided difference by

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

$$= \frac{\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

where $f[x_0]$, $f[x_1, x_0]$, *and* $f[x_2, x_1, x_0]$ *are called bracketed functions of their variables enclosed in square brackets.*

Rewriting,

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

This leads us to writing the general form of the Newton's divided difference polynomial for $n+1$ *data points,* $(x_0, y_0), (x_1, y_1), \ldots\ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)$, *as*

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots. + b_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

$$b_{n-1} = f[x_{n-1}, x_{n-2}, \ldots., x_0]$$

$$b_n = f[x_n, x_{n-1}, \ldots., x_0]$$

where the definition of the m^{th} *divided difference is*

$$b_m = f[x_m, \ldots\ldots\ldots, x_0]$$

$$= \frac{f[x_m, \ldots\ldots\ldots, x_1] - f[x_{m-1}, \ldots\ldots\ldots, x_0]}{x_m - x_0}$$

From the above definition, it can be seen that the divided differences are calculated recursively.

For an example of a third order polynomial, given $(x_0, y_0),\ (x_1, y_1),\ (x_2, y_2),$ *and* $(x_3, y_3),$

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) \\ + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$

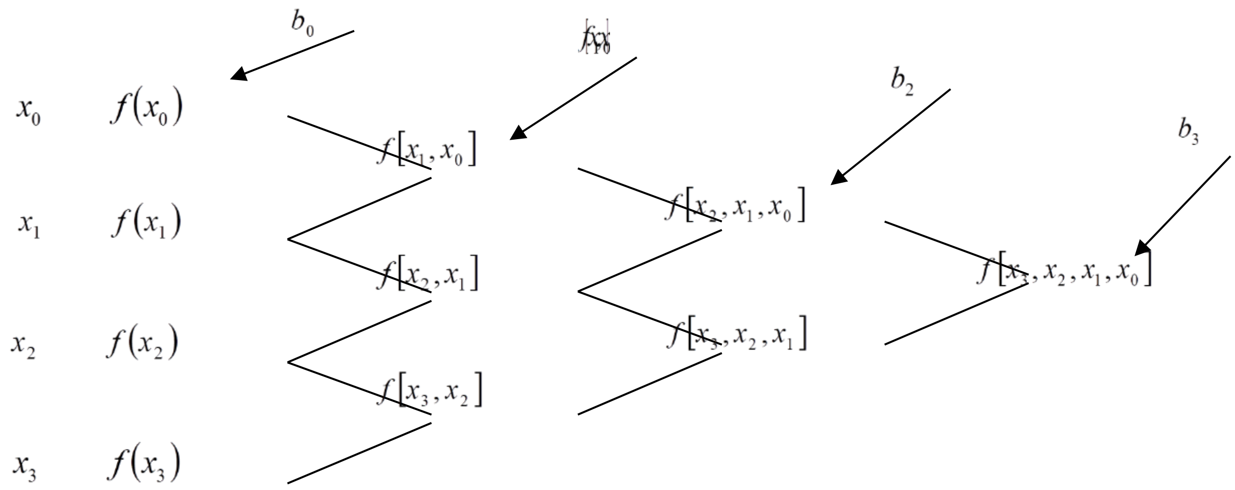

Figure 5 *Table of divided differences for a cubic polynomial.*

Example 6

Given the values

x:	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202

evaluate $f(9)$, using Newton's divided difference formula

Solution:

The divided differences table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
5	150	$\frac{392-150}{7-5}=121$		
7	392		$\frac{265-121}{11-5}=24$	
		$\frac{1452-392}{11-7}=265$		$\frac{32-24}{13-5}=1$
11	1452		$\frac{457-265}{13-7}=32$	
		$\frac{2366-1452}{13-11}=457$		$\frac{42-32}{17-7}=1$
13	2366		$\frac{709-457}{17-11}=42$	
		$\frac{5202-2366}{17-13}=709$		
17	5202			

Taking $x = 9$ in the Newton's divided difference formula, we obtain

$$f(9) = 150 + (9-5) \times 121 + (9-5)(9-7) \times 24 + (9-5)(9-7)(9-11) \times 1$$

$$= 150 + 484 + 192 - 16 = 810.$$

Example 7

Using Newton's divided differences formula, evaluate $f(8)$ and $f(15)$ given:

x:	4	5	7	10	11	13
$y = f(x)$:	48	100	294	900	1210	2028

Solution:

The divided differences table is

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
4	48				0
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		0
		310		1	
11	1210		33		
		409			
13	2028				

Taking $x = 8$ in the Newton's divided difference formula, we obtain

$$f(8) = 48 + (8 - 4)\,52 + (8 - 4)(8 - 5)\,15 + (8 - 4)(8 - 5)(8 - 7)\,1$$
$$= \mathbf{448.}$$

Similarly $f(15) = 3150$.

Example 8

Determine $f(x)$ as a polynomial in x for the following data:

x:	-4	-1	0	2	5
$y = f(x)$:	1245	33	5	9	1335

Solution:

The divided differences table is

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-4	1245				
		-404			
-1	33		94		
		-28		-14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

Applying Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \cdots$$
$$= 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94)$$
$$+ (x + 4)(x + 1)(x - 0)(-14) + (x + 4)(x + 1)x(x - 2)(3)$$
$$= 3x^4 - 5x^2 + 6x^2 - 14x + 5$$

Newton's Interpolation Formula for Equal Intervals

Let the function $y = f(x)$ take the values $y_0, y_1, \dots, y_n$ corresponding to the values $x_0, x_1, \dots, x_n$ of x. Let these values of x be equispaced such that $x_i = x_0 + ih$ $(i = 0,1, \dots.)$. Assuming $y(x)$ to be a polynomial of the nth degree in x such that $y(x_0) = y_0, y(x_1) = y_1, \dots, y(x_n) = y_n$. We can write

$$y(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2)$$
$$+ \cdots + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \qquad (1)$$

Putting $x = x_0, x_1, \cdots, x_n$ successively in (1), we get

$$y_0 = a_0, y_1 = a_0 + a_1(x_1 - x_0), y_2 = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

and so on.

From these, we find that $a_0 = y_0, \Delta y_0 = y_1 - y_0 = a_1(x_1 - x_0) = a_1 h$

$$\therefore \quad a_1 = \frac{1}{h}\Delta y_0$$

Also $\quad \Delta y_1 = y_2 - y_1 = a_1(x_2 - x_1) + a_2(x_2 - x_0)(x_2 - x_1)$

$$= a_1 h + a_2 hh = \Delta y_0 + 2h^2 a_2$$

$$\therefore \quad a_2 = \frac{1}{2h^2}(\Delta y_1 - \Delta y_0) = \frac{1}{2!h^2}\Delta^2 y_0$$

Similarly $a_3 = \frac{1}{3!h^3}\Delta^3 y_0$ and so on.

Substituting these values in (1), we obtain

$$y(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + \cdots \qquad (2)$$